Experimental Uncertainty

ME 242 Mechanical Engineering Systems

MISSOURI S&T

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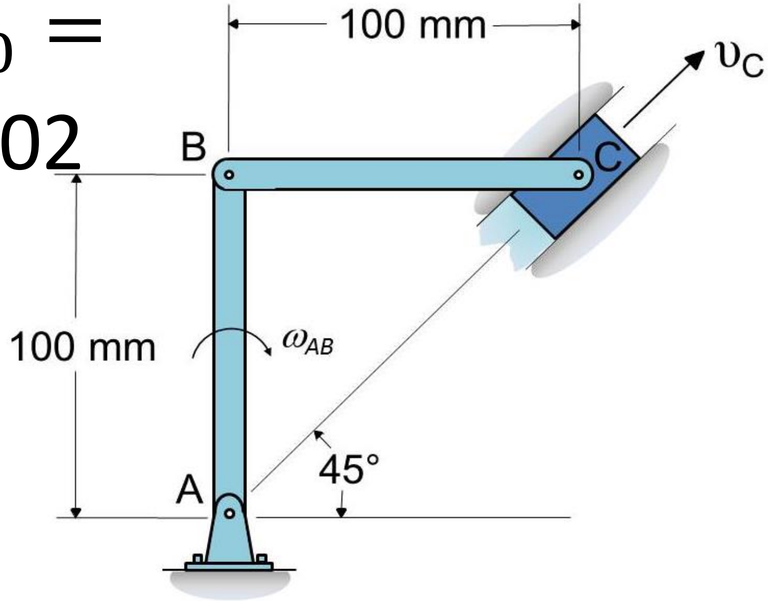
MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

Outline

- Experimental Error
- Types of Experimental Error
- Statistics
- Normal Distribution
- Combine uncertainty
- Uncertainty propagation
- Conclusions

An Introductory Example

- A dynamics problem: given $\omega_{AB} = 4$ rad/s, find v_c
- ω_{AB} may be measured.
- Let $X = \omega_{AB}$
- Ten measurements: $(X_i)_{i=1,10} =$
- (3.96 3.99 4.02 4.01 3.98 4.02 3.97 3.99 3.98 4.0) rad/s
- Uncertainty exists.



Measurement Error

- Measurement error is the difference between the measured value of a quantity and its true value.
- Measurement error is unavoidable and can be estimated.
- The measurement can be written as
  - $X = \bar{X} \pm U$
  - $\bar{X}$ is the best estimate
  - U is the uncertainty term
  - The true value may be between $\bar{X} - U$ and $\bar{X} + U$.
  - If U has 95% coverage (confidence), it is called the expanded uncertainty by ASME.

Types of the Error

Type	Examples	How to minimize it
Random errors • are caused by unknown and unpredictable changes	• The previous ten different measurements of the angular velocity	• Can be reduced by averaging over more observations
Systematic errors • occur when the equipment is improperly constructed, calibrated or used. • always occurs with the same value when use the instrument in the same way	• Tape measure has been stretched out from years of use • A stopwatch is accurate around 20°C, but you use it at 40°C.	• Hard to detect and to eliminate • The instrument maker may provide the estimate • Calibrate the instrument

How Do We Model Uncertainty?

- Measure $X = \omega_{AB}$ ten times, we get
- $X =$ (3.96 3.99 4.02 4.01 3.98 4.02 3.97 3.99 3.98   4.0) rad/s
- How do we use the samples?

- Average $\bar{X} = \frac{1}{10}$(3.96+3.99+ $\cdots +$ 3.98+4.0)
=$\frac{1}{10}\sum_{i=1}^{10} X_i = 3.992$ rad/s = 3.99 rad/s

How Do We Measure the Dispersion?

- $X = (3.96\ \ 3.99\ \ 4.02\ \ 4.01\ 3.98\ \ 4.02\ 3.97\ \ 3.99\ \ 3.98\ \ \ 4.0)$
- We could use $X_i - \bar{X}$ and $\frac{1}{N}\sum(X_i - \bar{X})$, $N = 10$
- But $\frac{1}{N}\sum(X_i - \bar{X})=0$.
- To avoid 0, we use $\frac{1}{N}\sum(X_i - \bar{X})^2$; to have the same unit as $\bar{X}$, we use $\sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2}$
- We actually use

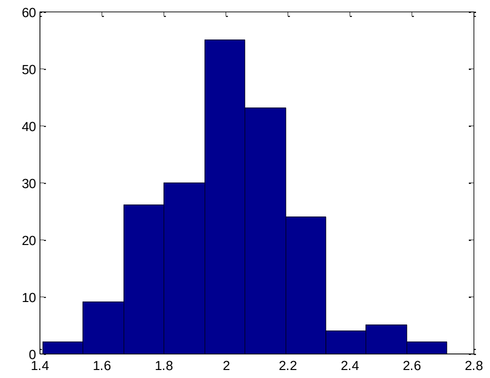
  Standard deviation: $s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}$.
- s is called the standard uncertainty by ASME.
- We found $s = 0.0204 = 0.02$ rad/s.

More about Standard Deviation (std)

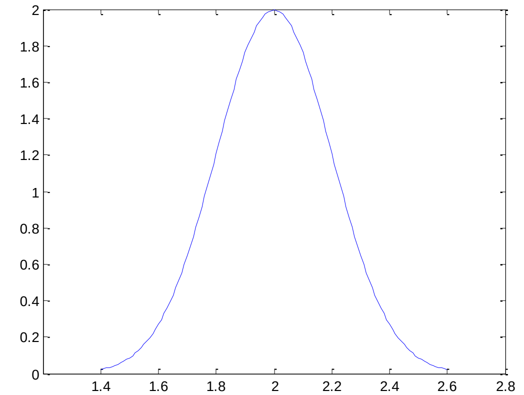
- It indicates how data spread around the mean.
- It is always non-negative.
- High std means
  - High dispersion
  - High uncertainty

Probability Distribution

- With more samples, we can draw a histogram.

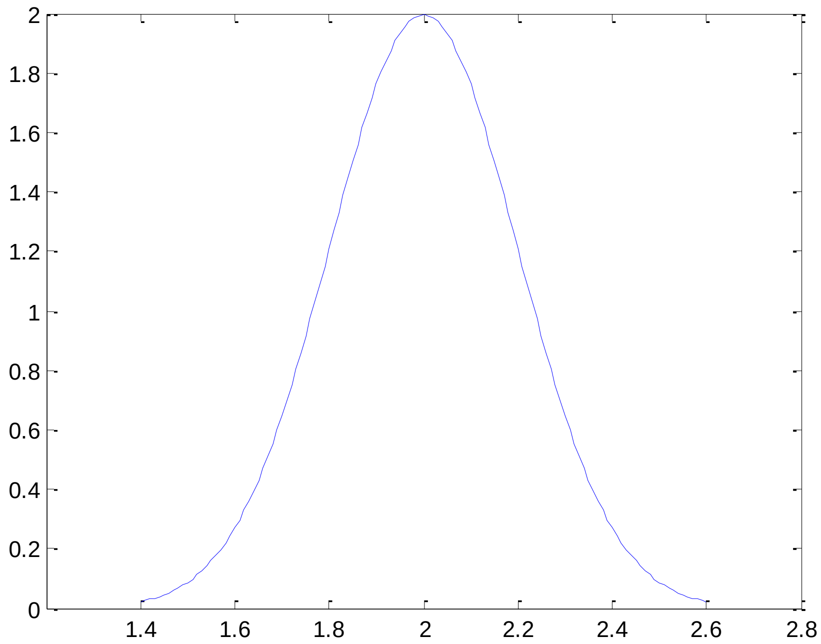


- If y-axis = frequency/(width of the bins in x-axis) and the number of samples is infinity, we get a probability density function (PDF) $f(x)$.



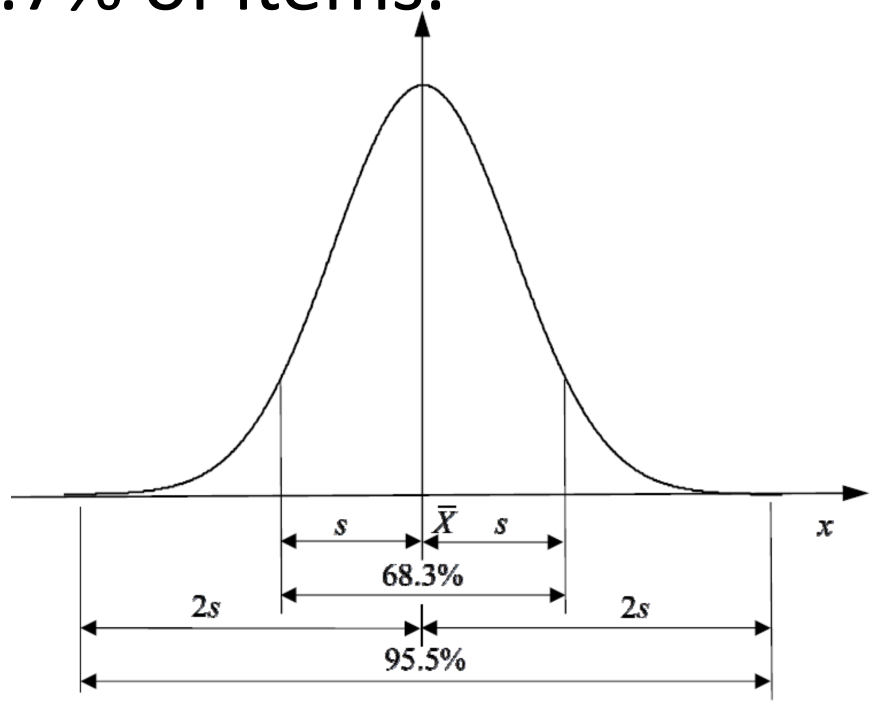
Normal Distribution

- $X \sim N(\overline{X}, s^2)$



More about Normal Distribution

- $\bar{X} \pm s$ contains 68.3% of items.
- $\bar{X} \pm 2s$ contains 95.5% of items.
- $X \pm 3s$ contains 99.7% of items.



Angular Velocity Example

- Using a normal distribution, report the measurement result at the 95% confidence level?
- Since $\bar{X} \pm 2s$ has a 95% coverage, we use $2s$.
  - Best estimate $= \bar{X} = 3.99$ rad/s
  - Uncertainty $U = 2s = 2(0.02) = 0.04$ rad/s
  - $X = \bar{X} \pm U$
  - $X = \omega_{AB} = 3.99 \pm 0.04$ rad/s

Given

$X = \omega_{AB} =$ (3.96 3.99 4.02 4.01 3.98 4.02 3.97 3.99 3.98 4.0) rad/s

What we found

$\bar{X} = 3.99$ rad/s

s =0.02 rad/s

More about 95% Confidence

- The range $X \in [3.99 - 0.04, 3.99 + 0.04]$in the example is called the 95% confidence interval.
- The likelihood of the interval covers the true value is 95%.
- We expect that there is only one chance in 20 that the true value does not lie within the specified range.

Two Random Variables

- X_i with $\bar{X}_i$ and s_i
- X_i $(i = 1,2)$ are independent
- $Y = X_1 + X_2$
- $\bar{Y} = \bar{X}_1 + \bar{X}_2$
- $s_Y = \sqrt{s_1^2 + s_2^2}$

Combined Uncertainty

- There are two independent sources of uncertainty
- Total error = error 1 + error 2
- U_1- Uncertainty from source 1, $U_1 = 2s_1$
- U_2- Uncertainty from source 2, $U_2 = 2s_2$
- $s = \sqrt{s_1^2 + s_2^2}$
- Combined uncertainty $U = 2s$
- Or $U = \sqrt{U_1^2 + U_2^2}$
- The result can be extended to more than two sources of uncertainty.

The Angular Velocity Example

- $X = \omega_{AB} =$ (3.96 3.99 4.02 4.01 3.98 4.02 3.97 3.99 3.98 4.0) rad/s
- Other source: The measuring device manufacturer claims an accuracy of ± 0.03 rad/s readout at 95% confidence level.
- What we've found
  - $\bar{X} = 4.0$ rad/s
  - $s_1 =$0.02 rad/s and $U_1 =$0.04 rad/s
- Now $U_2 =$0.03 rad/s
- Combined uncertainty $U = \sqrt{0.04^2 + 0.03^2} = 0.05$ rad/s
- Then $\omega_{AB} = X = 4.0 \pm 0.05$ rad/s

Uncertainty Propagation

- If Y is a function of X_i $(i = 1,2,\cdots,n)$
- $Y = f(X_1, X_2, \ldots, X_n)$
- X_i is measured as $\bar{X}_i \pm U_i$
- What is Y or $Y = \bar{Y} \pm U_Y$?

A Linear Function

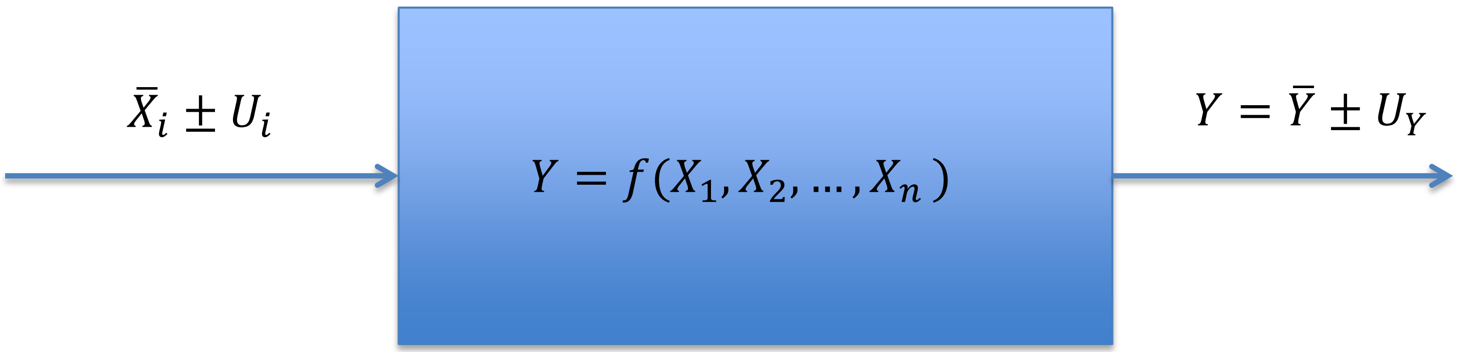
- $X_i \ (i = 1,2,\cdots,n)$ are independent
- $Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$
- $c_i \ (i = 0,1,2,\cdots,n)$ are constant.
- Then $\bar{Y} = c_0 + c_1 \bar{X}_1 + c_2 \bar{X}_2 + \cdots + c_n \bar{X}_n$
- $s_Y = \sqrt{c_1^2 s_1^2 + c_2^2 s_2^2 + \cdots + c_n^2 s_n^2}$

A Nonlinear Function

- $Y = f(X_1, X_2, \dots, X_n)$
- $\bar{\boldsymbol{X}} = f(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)$
- Taylor expansion series
- $Y \approx c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$
- $c_i = \frac{\partial f}{\partial X_i}$ at $\bar{\boldsymbol{X}}$ $(i = 1,2,\cdots,n)$
- $\bar{Y} = f(\bar{\boldsymbol{X}})$
- $s_Y = \sqrt{c_1^2 s_1^2 + c_2^2 s_2^2 + \cdots + c_n^2 s_n^2}$

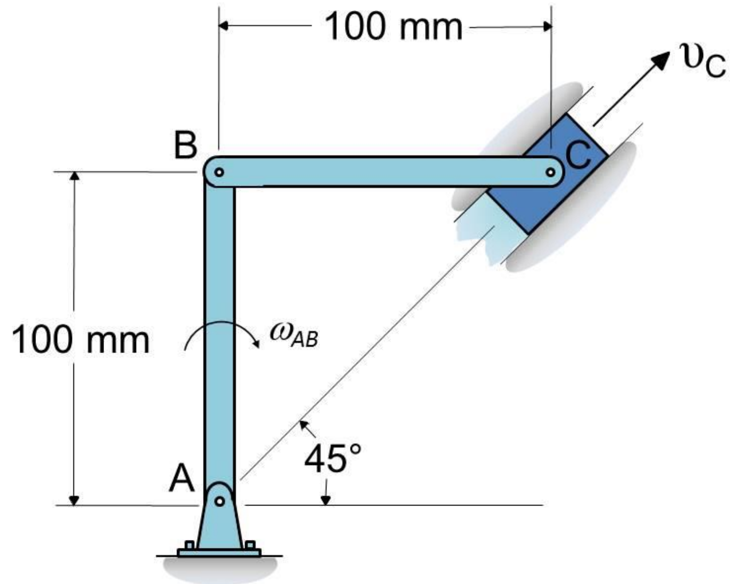
Estimate of Y

- $Y = \bar{Y} \pm U_Y$
- $U_Y = 2s_Y$



Example

- Given $\omega_{AB} = 3.99 \pm 0.05$ rad/s, $L_{AB} = 100.0 \pm 0.1$ mm
- Find v_c
- Solution
  - $X_1 = \omega_{AB}, \bar{X}_1 = 4$ rad/s, $U_1 = 0.05$ rad/s
  - $X_2 = L_{AB}, \bar{X}_2 = 100$ m, $U_2 = 0.1$ mm
  - From dynamics
  - $Y = v_c = \frac{L_{AB}\omega_{AB}}{\cos 45^\circ} = \frac{X_1 X_2}{\cos 45^\circ}$
  - $\bar{Y} = \frac{100(3.99)}{\cos 45^\circ} = 564.3$ mm/s

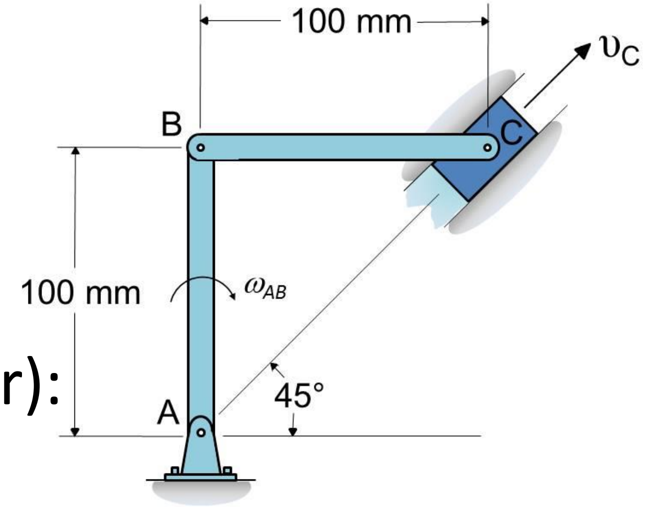


- $c_1 = \frac{\partial f}{\partial X_1} = \frac{X_2}{\cos 45^\circ} = \frac{100}{\cos 45^\circ} = 141.42$
- $c_2 = \frac{\partial f}{\partial X_2} = \frac{X_1}{\cos 45^\circ} = \frac{3.99}{\cos 45^\circ} = 5.64$
- $U_Y = \sqrt{c_1^2 U_1^2 + U_2^2 \sigma_2^2} = \sqrt{141.42^2(0.05)^2 + 5.64^2(0.1)^2}$
  $= 7.09 = 7.1$ mm/s

Result

- $v_c = 564.3 \pm 7.1$ mm/s
- The confidence is approximately 95%.

Review What We've Done



- Measurement of $X_1 = \omega_{AB}$
  - Error source 1: (3.96 3.99 4.02 4.01 3.98 4.02 3.97 3.99 3.98 4.0) rad/s
  - $\bar{X}_1 = 3.99$ rad/s, $U_1 = 0.04$ rad
  - Error source 2 (from the device maker): $U_2 = 0.03$ rad
  - Combined uncertainty $U = \sqrt{0.04^2 + 0.03^2} = 0.05$ rand/s
- Measurement of $X_2 = L_{AB} = 100 \pm 0.01$ mm
- Uncertainty propagation for estimating v_c
  - $v_c = 564.3 \pm 7.1$ mm/s

Conclusions

- Measurements of any physical quantity may never be exact.
- We only know its value with a range of uncertainty.
- the measurement can be written as
  - $\bar{X} \pm U$
  - The true value may be between $X = \bar{X} - U$ and $\bar{X} + U$ with a certain confidence
- The uncertainty U can by qualified with the approaches presented in this lecture.